BHABHA SCATTERING

Abstract

The electron charge is considered to be distributed or extended in space. The differential of the electron charge is set equal to a function of electron charge coordinates multiplied by a four-dimensional differential volume element. The four-dimensional integral of this function is required to equal the electron charge in all Lorentz frames. The positron is also considered to be distributed or extended in space, and is treated similarly. The probability amplitude for scattering of such an electron and such a positron is calculated when there is a single photon exchanged between the two particles. A second probability amplitude is calculated where the electron and the positron annihilate producing a photon. The photon then annihilates producing an electron-positron pair. Each probability amplitude is the probability amplitude for point particles multiplied by two form factors-one for each particle. The S-matrix is the difference between the two probability amplitudes.

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I. INTRODUCTION

In the rest frame of an electron charge distribution, let $x_r'^{\nu} = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ denote a spacetime charge point, and let $x_r^\nu = (x_r^0, x_r^1, x_r^2, x_r^3)$ denote the center of the charge distribution. Sometimes the superscript on the four-vector (not the components) will be omitted, and we will write $x_r' = (x_r'^0, x_r'^1, x_r'^2, x_r'^3)$ and $x_r = (x_r^0, x_r^1, x_r^2, x_r^3)$. Introduce $\tilde{x}_r = x_r' - x_r$ or equivalently $\tilde{x}_r^\nu = x_r'^\nu - x_r^\nu$. In a frame of reference in which the electron charge distribution moves with a speed β in the $+x^3$ direction, let $x_m' = (x_m'^0, x_m'^1, x_m'^2, x_m'^3)$ denote a spacetime charge point, and let $x_m = (x_m^0, x_m^1, x_m^2, x_m^3)$ denote the center of the charge distribution. Introduce $\tilde{x}_m = x_m' - x_m$. A Lorentz transformation yields $\tilde{x}_r^1 = \tilde{x}_m^1$, $\tilde{x}_r^2 = \tilde{x}_m^2$, $\tilde{x}_r^3 = \gamma(\tilde{x}_m^3 - \beta\tilde{x}_m^0)$, and $\tilde{x}_r^0 = \gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)$ where $\gamma = 1/\sqrt{1-\beta^2}$. Denote this Lorentz transformation by $\tilde{x}_r = L(\tilde{x}_m)$.

In the rest frame, the electron charge e is equal to $\int \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r$ where $\rho_r(\tilde{x}_r)$ is the charge density in the rest frame and δ denotes the delta function.[1] So an element of charge in the rest frame is given by

$$de_r = \rho_r(\tilde{x}_r)\delta(\tilde{x}_r^0)d^4\tilde{x}_r. \tag{1}$$

In the m frame, the electron charge e is equal to $\int \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}_m^0 - \beta\tilde{x}_m^3)]d^4\tilde{x}_m$.[2] So an element of charge de_m in the m frame is given by

$$de_m = \rho_r(L(\tilde{x}_m))\delta[\gamma(\tilde{x}^0_m - \beta\tilde{x}^3_m)]d^4\tilde{x}_m. \qquad (2)$$

However Eq. (2) will not be needed because of invariance of the form factors.

The next section is a review of point electron-point positron scattering, which is known as Bhabha scattering. The third section will study extended electron and extended positron scattering. As mentioned in the abstract, there are two probability amplitudes to calculate. The S-matrix is the difference between these two probability amplitudes. A short discussion follows.

II. ELECTRON-POSITRON SCATTERING

The calculation for electron-positron scattering will follow Bjorken and Drell.[3] When the point electron at spacetime point x and point positron at spacetime point y exchange a single photon, the probability amplitude is approximated by

$$S_{fi1} = -\int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma^\nu)\phi_i(x)iD_f(x-y)\bar{\phi}_F(y)(-ie\gamma_\nu)\phi_I(y). \qquad (3)$$

The initial exact electron wave function is approximated by the plane wave solution to the Dirac equation. The plane wave solution, which is normalized to unity in a box of volume V, is

$$\phi_i(x) = \sqrt{\frac{m}{E_i V}}\, u_i \exp\left(-ip_i \cdot x\right) \tag{4}$$

where $\hbar$ and c have been set equal to 1, m is the electron rest mass, u_i is a four-component spinor, which depends on the initial spin and on $p_i = (p_i^0 = E_i, p_i^1, p_i^2, p_i^3)$, the initial four-momentum, and γ^ν are the four Dirac matrices, which are labelled by $\nu = 0, 1, 2, 3$. The final electron wave function is

$$\phi_f(x) = \sqrt{\frac{m}{E_f V}}\, u_f \exp\left(-ip_f \cdot x\right), \tag{5}$$

where p_f is the final electron four-momentum, E_f is the final electron energy, u_f is the final electron spinor, and $\bar{\phi}_f = \phi_f^\dagger \gamma^0$. The photon propagator $D_F(x-y)$ is given by

$$D_F(x-y) = \int \frac{d^4q}{(2\pi)^4} \exp\left[-iq \cdot (x-y)\right] \frac{-1}{q^2 + i\epsilon} \tag{6}$$

where q is the four-momentum of the photon.

Let $p_I = (E_I, p_I^1, p_I^2, p_I^3)$ denote the initial positron four-momentum, and let $p_F = (E_F, p_F^1, p_F^2, p_F^3)$ denote the final positron four-momentum.

It is common to treat the positive energy positron, which propagates forward in time, as a negative energy electron, which propagates backward in time.[4] Thus the negative energy electron propagates from the future into the past. To accommodate this, the initial four-momentum of this electron is identified as $-p_F = (-E_F, -\mathbf{p}_F)$, and the final four-momentum of this electron is identified as $-p_I = (-E_I, -\mathbf{p}_I)$. Also note that the spacetime point y is now the argument of the negative energy electron wave function. So the initial plane wave function of this electron, which has been normalized to unity in a box of volume V, now is

$$\phi_I(y) = \sqrt{\frac{m}{E_F V}}\, v_F \exp\left(+ip_F \cdot y\right), \tag{7}$$

and the final plane wave function is

$$\phi_F(y) = \sqrt{\frac{m}{E_I V}}\, v_I \exp\left(+ip_I \cdot y\right) \tag{8}$$

Here v_F is a one by four column matrix, which depends on the final positron spin and on the final positron four-momentum, and v_I is a one by four column matrix, which depends on the initial positron spin and on the initial four-momentum.

Substituting Eqs. (4), (5), (6), (7), and (8) into Eq. (3) yields

$$S_{fi1} = \frac{-ie^2 m^2 (\bar{u}_f \gamma^\nu u_i)(\bar{v}_I \gamma_\nu v_F)}{(2\pi)^4 V^2 \sqrt{E_i E_f E_I E_F}} \int \frac{d^4x\, d^4y\, d^4q\, \exp[i(p_f - p_i - q)\cdot x]\exp[i(p_F - p_I + q)\cdot y]}{q^2 + i\epsilon}. \quad (9)$$

Perform the following integrations:

$$\int \exp(i(p_f - p_i - q)\cdot x) d^4x = (2\pi)^4 \delta^4(p_f - p_i - q); \quad (10)$$

$$\int \exp(i(p_F - p_I + q)\cdot y) d^4y = (2\pi)^4 \delta^4(p_F - p_I + q); \quad (11)$$

$$\int \delta^4(p_f - p_i - q)\delta^4(p_F - p_I + q)\frac{d^4q}{q^2 + i\epsilon} = \frac{\delta^4(p_F + p_f - p_I - p_i)}{(p_f - p_i)^2}; \quad (12)$$

and find

$$S_{fi1} = \frac{-ie^2 mM}{V^2 \sqrt{E_i E_f E_I E_F}} (2\pi)^4 \delta^4(p_f + p_F - p_i - p_I) \frac{(\bar{u}_f \gamma^\nu u_i)(\bar{v}_I \gamma_\nu v_F)}{(p_f - p_i)^2}. \quad (13)$$

Next, calculate the second probability amplitude where the electron and the positron annihilate at spacetime point y producing a photon, which propagates to spacetime point x, where the photon produces an electron-positron pair. The probability amplitude for this process is

$$S_{fi2} = -\int d^4x\, d^4y\, \bar{\phi}_f(x)(-ie\gamma^\nu)\phi_I(x) iD_F(x-y)\bar{\phi}_F(y)(-ie\gamma_\nu)\phi_i(y). \tag{14}$$

Substituting Eqs. (4), (5), (6), (7), and (8) into Eq. (14) yields

$$S_{fi2} = \frac{-ie^2 mM(\bar{u}_f\gamma^\nu v_F)(\bar{v}_I\gamma_\nu u_i)}{(2\pi)^4 V^2\sqrt{E_i E_f E_I E_F}} \int \frac{d^4x\, d^4y\, d^4q\, \exp[i(p_f + p_F - q)\cdot x]\exp[i(-p_i - p_I + q)\cdot y]}{q^2 + i\epsilon}, \tag{15}$$

and finally

$$S_{fi2} = \frac{-ie^2 mM}{V^2\sqrt{E_i E_f E_I E_F}}(2\pi)^4\delta^4(p_f + p_F - p_i - p_I)\frac{(\bar{u}_f\gamma^\nu v_F)(\bar{v}_I\gamma_\nu u_i)}{(p_i + p_I)^2}. \tag{16}$$

The S-matrix is $S_{fi} = S_{fi1} - S_{fi2}$.

III. EXTENDED ELECTRON-POSITRON SCATTERING

Suppose that the electron initially moves with a speed β in the $+x^3$ direction of the m frame. Recall that x'_m is a spacetime electron charge point, x_m is the center of the electron charge distribution and also the argument of the electron wave function. Define $\tilde{x}_m = x'_m - x_m$. Recall that an element of electron charge in the rest frame is labeled de_r, and is given by Eq. (1). Also recall that an element of electron charge in the m frame is labeled de_m.

In the rest frame of the positron, let y'_r be a spacetime charge point, and let y_r be the center of the positron charge. Introduce $\tilde{y}_r = y'_r - y_r$. An element of the positron charge in its rest frame, i.e. an element of the negative energy electron charge in that rest frame, is given by Eq. (1) with $\tilde{x}_r$ replaced by $\tilde{y}_r$. Take the positron to be moving with a speed β in the $-y^3$ direction in the m frame. The m frame is therefore the center of mass frame. Let y'_m be a spacetime charge point of the positron. Let y_m be the center of the positron charge and the argument of its wave function. Introduce $\tilde{y}_m = y'_m - y_m$. The interaction takes place at charge points, so replace $D_F(x - y)$ by $D_F(x'_m - y'_m)$ in Eq. (14). Also, replace the positive energy electron charge by the four-dimensional integral of de_m, and replace the negative energy charge by the four-dimensional integral of de_m (the speeds are the same). So now the first probability amplitude S_{FI1} is

$$S_{FI1} = -\int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-i\gamma^\nu)\phi_i(x_m)de_m$$
$$iD_F(x'_m - y'_m)\bar{\phi}_F(y_m)(-i\gamma_\nu)\phi_I(y_m)de_m. \quad (17)$$

The four-momentum of the photon in the m frame is labeled q_m. Use $D_F(x'_m - y'_m) = D_F(x_m - y_m)\exp(-i\tilde{x}_m \cdot q_m)\exp(+i\tilde{y}_m \cdot q_m)$ to show

$$S_{FI1} = -\int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-ie\gamma^\nu)\phi_i(x_m)iD_F(x_m - y_m)\bar{\phi}_F(y_m)(-ie\gamma_\nu)\phi_I(y_m)$$
$$\int \exp(-i\tilde{x}_m \cdot q_m)\frac{de_m}{e}\int \exp(-i\tilde{y}_m \cdot q_m)\frac{de_m}{e} \quad (18)$$

By the delta function which results when integrating the first integral above, $q_m = p_f - p_i$ where p_f and p_i are measured in the m frame. By Eq. (3), the first integral is S_{fi1}, so

$$S_{FI1} = S_{fi1}F_1(q)F_2(q) \quad (19)$$

where

$$F_1(q) = \int \exp(-i\tilde{x}_m \cdot q_m)\frac{de_m}{e} \quad (20)$$

is the positive energy electron form factor and

$$F_2(q) = \int \exp(+i\tilde{y}_m \cdot q_m)\frac{de_m}{e} \tag{21}$$

is the negative energy electron form factor. Due to invariance of the form factor,[2]

$$F_1(q) = \int \exp(-i\tilde{x}_r \cdot q_r)\frac{\rho_r(\tilde{x}_r)}{e}\delta(\tilde{x}_r^0)d^4\tilde{x}_r \tag{22}$$

and

$$F_2(q) = \int \exp(+i\tilde{y}_r \cdot q_r)\frac{\rho_r(\tilde{y}_r)}{e}\delta(\tilde{y}_r^0)d^4\tilde{y}_r\,. \tag{23}$$

To get a rough idea of how size and structure affect scattering, pick the electron and positron charge to be uniformly distributed on a spherical shell of radius a in their respective rest frames. Then the charge densities will be given by $\rho_r(\tilde{x}_r) = e\delta(|\tilde{\mathbf{x}}_r| - a)/4\pi\, a^2$ and $\rho_r(\tilde{y}_r) = e\delta(|\tilde{\mathbf{y}}_r| - a)/4\pi\, a^2$. The results are $F_1(q) = F_2(q) = j_0(|\mathbf{q}_r|a)$ where j_0 is the spherical Bessel function of order zero and

$$|\mathbf{q}_r|^2 = (q_m^1)^2 + (q_m^2)^2 + \gamma^2(q_m^3 - \beta q_m^0)^2 = \frac{(p_f \cdot p_i)^2 - m^4}{m^2}.^2 \tag{24}$$

For relativistic speeds, the electron mass can be neglected, so $E_i = |\mathbf{p}_i|$ and $E_f = |\mathbf{p}_f|$. In the center of mass frame, $|\mathbf{p}_f| = |\mathbf{p}_i|$, so $E_f = E_i$. Then

$$|\mathbf{q}_r|^2 \approx \frac{[E_f E_i(1-\cos(\theta))]^2}{m^2} = \frac{4E_f^2 E_i^2 \sin^4(\theta/2)}{m^2} \tag{25}$$

Put back $\hbar$ and c in Eq. (25), use $|\mathbf{p_i}| \approx \gamma mc$ and find

$$|\mathbf{q}_r| \approx \frac{2(mc)\sin^2(\theta/2)}{\hbar(1-\beta^2)}. \tag{26}$$

The second probability amplitude, S_{FI2}, is given by

$$S_{FI2} = -\int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-ie\gamma^\nu)\phi_I(x_m)\frac{de_m}{e}$$
$$iD_F(x'_m - y'_m)\bar{\phi}_F(y_m)(-ie\gamma_\nu)\phi_i(y_m)\frac{de_m}{e}. \tag{27}$$

This equation can be written as

$$S_{FI2} = -\int d^4x_m\, d^4y_m\, \bar{\phi}_f(x_m)(-ie\gamma^\nu)\phi_I(x_m)iD_F(x_m - y_m)\bar{\phi}_F(y_m)(-ie\gamma_\nu)\phi_i(y_m)$$
$$\int \exp(-i\tilde{x}_m \cdot q'_m)\frac{de_m}{e} \int \exp(+i\tilde{y}_m \cdot q'_m)\frac{de_m}{e}. \tag{28}$$

By the delta function which results when integrating the first integral above, $q'_m = p_f + p_i$. Integration leads to

$$S_{FI2} = S_{fi2} F'_1(q) F'_2(q)\,. \tag{29}$$

where by invariance of the form factor

$$F_1'(q) = \int \exp(-i\tilde{x}_r \cdot q_r') \frac{\rho_r(\tilde{x}_r)}{e} \delta(\tilde{x}_r^0) d^4\tilde{x}_r \tag{30}$$

and

$$F_2'(q) = \int \exp(+i\tilde{y}_r \cdot q_r') \frac{\rho_r(\tilde{y}_r)}{e} \delta(\tilde{y}_r^0) d^4\tilde{y}_r \tag{31}$$

For charge of both particles distributed uniformly on a spherical shell of radius a, $F_1'(q) = F_2'(q) = j_0(|\mathbf{q}_r'|a)$. Repeating a previous calculation,

$$|\mathbf{q}_r'|^2 = ({q_m'}^1)^2 + ({q_m'}^2)^2 + \gamma^2({q_m'}^3 - \beta {q_m'}^0)^2 = \frac{(p_F \cdot p_i)^2 - m^4}{m^2}.^2 \tag{32}$$

Note that $|\mathbf{p}_f| = |\mathbf{p}_i| = E_F = E_i$ for relativistic speeds in the center of mass frame. Since $\mathbf{p}_F$ is in the opposite direction as $\mathbf{p}_f$,

$$|\mathbf{q}_r|^2 \approx \frac{[E_f E_i(1 + \cos(\theta))]^2}{m^2} = \frac{4E_f^2 E_i^2 \cos^4(\theta/2)}{m^2} \tag{33}$$

The S-matrix $S_{FI} = S_{FI1} - S_{FI2}$.

IV. DISCUSSION

As previously discussed, there are two probability amplitudes in Bhabha scattering. For the extented electron, each probability amplitude is multiplied by an electron form factor and a positron form factor. The S-matrix is the difference between the two probability amplitudes..

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